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A NEW APPROACH FOR TWO-DIMENSIONAL NONLINEAR MIXED VOLTERRA-FREDHOLM INTEGRAL EQUATIONS AND ITS CONVERGENCE ANALYSIS

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Abstract. In this article, a fast numerical scheme is investigated to approximate nonlinear mixed Volterra-Fredholm integral equations based on expansion method. In the approximation procedure, we use expansion method in order to transform these two-dimensional integral equations into a differential equation. After making boundary conditions, this differential equation decreases to a system algebraic equations that can be solved simply using any of the common methods. The main characteristic of this scheme is low computational cost and CPU time to achieve an appropriate solution. Error analysis and comparisons with other existing schemes demonstrate the efficiency of the proposed scheme.

Keywords: two-dimensional expansion, linear and nonlinear systems, mixed Volterra-Fredholm integral equations, boundary conditions.

AMS Subject Classification: 65R20,45B05.

1. Introduction

Let us consider general two-dimensional linear and nonlinear mixed Volterra-Fredholm integral equations of the second kind of the forms:

$$f(x,y) = g(x,y) + \lambda \int_a^y \int_a^b k(x,y,s,t) f(s,t)\, \mathrm{d}s\mathrm{d}t \qquad (x,y) \in I, \tag{1}$$

$$f(x,y) = g(x,y) + \lambda \int_a^y \int_a^b k(x,y,s,t) V\left(f(s,t)\right) \mathrm{d}s\mathrm{d}t \qquad (x,y) \in I. \tag{2}$$

Here λ is constant, $I = [a,b] \times [a,b]$, and $f(s,t)$ is unknown function, $g(x,y)$, $k(x,y,s,t)$ are continuous functions and $V\left(f(s,t)\right)$ is a nonlinear continuous function with respect to $f(x,y)$.

Existence and uniqueness results for (1) may be found in [13, 15, 28] (see also [16, 20] for the linear case).

Integral equations are often involved in various fields such as physics, engineering and biology problems, and hence they have been investigated extensively. For example, see [9, 8, 21, 30] and the references cited therein. Over the years, the integral equations and differential equations have been used increasingly in different areas of applied sciences. Mixed Volterra-Fredholm integral equations are extensively occurred in various fields in physics, engineering problems.

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Regarding that the essential features of the models are widely applicable in real world, so proposing a high-order numerical schemes is significant.

The essential properties of the models have wide range of applicability [10] numerical methods have been studied numerically for solving integral equations and mixed Volterra-Fredholm integral equations [3, 4, 5, 7, 12, 14, 27].

For example integral equations of the second type are solved with expansion method [18, 24, 29], with Gaussian radial basis function [2], with Wavelet [1, 22], with polynomial [6, 23] and with linearization method [11].

Also mixed Volterra-Fredholm integral equations are solved using Bernoulli collocation method, Bernstein polynomials, new iterative method and meshless methods [17, 19, 26, 27].

Here, we propose a numerical scheme based on a simple tool with high speed due to overcoming the difficulty of solving two-dimensional integral equations.

The outline of the paper: section 2 is devoted to solution of linear two-dimensional integral equation and section 3 is devoted to solution of nonlinear two-dimensional integral equation.

A expansion can be created for the solution $f(s,t)$ in the integral Eq. (1) - (2) as follows:

$$f(s,t)=\sum_{i=0}^{n}\sum_{j=0}^{n}\frac{1}{i!j!}\frac{\partial^i\partial^j}{\partial x^i\partial y^j}f(x,y)(s-x)^i(t-y)^j+E(s,t). \tag{3}$$

Here $E(s,t)$ denotes the error between $f(s,t)$ and its expansion Eq. (3) which can be written:

$$E(s,t)=\frac{1}{(n+1)!(n+1)!}\frac{\partial^{n+1}\partial^{n+1}}{\partial x^{n+1}\partial y^{n+1}}f(x,y)(s-x)^{n+1}(t-y)^{n+1}+\dots$$

2. Solution of linear two-dimensional integral equation

Consider the Eq. (1). By using the first n terms of Eq. (3) and neglecting the term $\int_a^y\int_a^b k(x,y,s,t)E(s,t)\mathrm{d}s\mathrm{d}t$ in Eq. (1), then by substituting Eq. (3) for $f(s,t)$ in the integral in Eq. (1), one can obtain:

$$f(x,y)-\lambda\sum_{i=0}^{n}\sum_{j=0}^{n}\frac{1}{i!j!}\frac{\partial^i\partial^j}{\partial x^i\partial y^j}f(x,y)\int_a^y\int_a^b k(x,y,s,t)(s-x)^i(t-y)^j\mathrm{d}s\mathrm{d}t\simeq g(x,y) \tag{4}$$

Thus, Eq. (4) is developed as a linear of partial differential equation that can be solved. However, this partial differential equation requires appropriate boundary circumstances.

To build boundary conditions, first both sides of Eq. (1) are.differentiated. So, we find the following differential equations:

$$\begin{cases}\frac{\partial f(x,y)}{\partial x}=\frac{\partial g(x,y)}{\partial x}+\lambda\frac{\partial}{\partial x}\int_a^b\int_a^b k(x,y,s,t)f(s,t)\mathrm{d}s\mathrm{d}t\\ \frac{\partial f(x,y)}{\partial y}=\frac{\partial g(x,y)}{\partial y}+\lambda\frac{\partial}{\partial y}\int_a^y\int_a^b k(x,y,s,t)f(s,t)\mathrm{d}s\mathrm{d}t\\ \quad\vdots\\ \frac{\partial^i\partial^j}{\partial x^i\partial y^j}f(x,y)=\frac{\partial^i\partial^j}{\partial x^i\partial y^j}g(x,y)+\lambda\frac{\partial^i\partial^j}{\partial x^i\partial y^j}\int_a^y\int_a^b k(x,y,s,t)f(s,t)\mathrm{d}s\mathrm{d}t.\end{cases} \tag{5}$$

Next, $f(s,t)$ is substituted by $f(x,y)$ to obtain for $i,j=1,\dots,n$

$$\frac{\partial^i\partial^j}{\partial x^i\partial y^j}f(x,y)\simeq\frac{\partial^i\partial^j}{\partial x^i\partial y^j}g(x,y)+\lambda\frac{\partial^i\partial^j}{\partial x^i\partial y^j}\int\limits_a^y\int\limits_a^b k(x,y,s,t)f(x,y)\mathrm{d}s\mathrm{d}t. \tag{6}$$

Now combination of Eq. (4) with Eq. (6) becomes a linear system algebraic equation that can be solved.

3. Solution of nonlinear two-dimensional integral equation

Consider the Eq. (2). If we use the first n terms of Eq. (3) and neglect the term $\int\limits_a^y\int\limits_a^b k(x,y,s,t)V\left(E(s,t)\right)\mathrm{d}s\mathrm{d}t$ in Eq. (2), then by substituting Eq. (3) for $f(s,t)$ in the integral in Eq. (2), we will get:

$$f(x,y)-\lambda\int\limits_a^y\int\limits_a^b k(x,y,s,t)V\left(\sum_{i=0}^{n}\sum_{j=0}^{n}\frac{1}{i!j!}\frac{\partial^i\partial^j}{\partial x^i\partial y^j}f(x,y)(s-x)^i(t-y)^j\right)\mathrm{d}s\mathrm{d}t\simeq g(x,y). \tag{7}$$

So Eq. (7) becomes a partial differential equation that can be solved. However, this partial differential equation requires appropriate boundary conditions.

To make boundary conditions, first both sides of Eq. (2) are differentiated. So, we find the following differential equations:

$$\begin{cases}\frac{\partial f(x,y)}{\partial x}=\frac{\partial g(x,y)}{\partial x}+\lambda\frac{\partial}{\partial x}\int\limits_a^y\int\limits_a^b k(x,y,s,t)V\left(f(s,t)\right)\mathrm{d}s\mathrm{d}t\\ \frac{\partial f(x,y)}{\partial y}=\frac{\partial g(x,y)}{\partial y}+\lambda\frac{\partial}{\partial y}\int\limits_a^y\int\limits_a^b k(x,y,s,t)V\left(f(s,t)\right)\mathrm{d}s\mathrm{d}t\\ \qquad\vdots\\ \frac{\partial^i\partial^j}{\partial x^i\partial y^j}f(x,y)=\frac{\partial^i\partial^j}{\partial x^i\partial y^j}g(x,y)+\lambda\frac{\partial^i\partial^j}{\partial x^i\partial y^j}\int_a^y\int\limits_a^b k(x,y,s,t)V\left(f(s,t)\right)\mathrm{d}s\mathrm{d}t.\end{cases} \tag{8}$$

Next, $f(s,t)$ is substituted by $f(x,y)$ to obtain for $i,j=1,\dots,n$

$$\frac{\partial^i\partial^j}{\partial x^i\partial y^j}f(x,y)\simeq\frac{\partial^i\partial^j}{\partial x^i\partial y^j}g(x,y)+\lambda\frac{\partial^i\partial^j}{\partial x^i\partial y^j}\int\limits_a^b\int\limits_a^b k(x,y,s,t)V\left(f(x,y)\right)\mathrm{d}s\mathrm{d}t. \tag{9}$$

Now combining of Eq. (7) with Eq. (9) becomes a nonlinear system algebraic equation that can be solved.

4. Convergence analysis

$$\|g(x,y)\|=\max|g(x,y)|_{\forall(x,y)\in I}.$$

We assume that $|k(x,y,s,t)|\leq M$ for all $(x,y),(s,t)\in I$ and let $Y=(b-a)(b-a)$ and M where is a positive of $2D$-Taylor is denoted by

$$e_{2D-T}=\|f_n(x,y)-f(x,y)\|.$$

Let $f(x,y)$ be an exact solution of Eq. (1) and $f_n(x,y)$ be the approximate solution of the Eq. (1). We present following theorems:

Theorem 4.1. *Suppose that $0<\alpha<1$ under the tacit assumptions above, the solution of Eq. (1), converges toward exact solution.*

Proof. Let

$$\begin{aligned}
\|f_n(x,y)-f(x,y)\| &= \max|f_n(x,y)-f(x,y)| \\
&= \max|g(x,y)+\Upsilon\lambda\int_0^y\int_0^1 k(x,y,s,t)f_n(x,y)\mathrm{d}s\mathrm{d}t \\
&- g(x,y)-\Upsilon\lambda\int_0^y\int_0^1 k(x,y,s,t)f(x,y)\mathrm{d}s\mathrm{d}t| \\
&\le \max|\lambda||\Upsilon|\int_0^y\int_0^1 |k(x,y,s,t)||f_n(x,y)-f(x,y)|\mathrm{d}s\mathrm{d}t \\
&\le |\lambda||\Upsilon|M\int_0^y\int_0^1 \max|f_n(x,y)-f(x,y)|\mathrm{d}s\mathrm{d}t \\
&= |\lambda||\Upsilon|M\|f_n(x,y)-f(x,y)\| \\
&\Rightarrow \|f_n(x,y)-f(x,y)\| \le \alpha\|f_n(x,y)-f(x,y)\|.
\end{aligned}$$

Where $\alpha = |\lambda||\Upsilon|M$. we get $(1-\alpha)\|f_n(x,y)-f(x,y)\| \le 0$ and choose $0<\alpha<1$ by increasing n, it implies $\|f_n(x,y)-f(x,y)\| \to 0$ as $n\to\infty$. □

Theorem 4.2. *Suppose that $0<\beta<1$ under the tacit assumptions above and suppose that the nonlinear term $V(f(s,t))$ is satisfied the Lipschitz condition*

$$|V(f(s,t))-V(u(s,t))| \le L|f(s,t)-u(s,t)|.$$

The solution of integral Eq. (2), converges toward exact solution.

Proof. Let

$$\begin{aligned}
\|f_n(x,y)-f(x,y)\| &= \max|f_n(x,y)-f(x,y)| \\
&= \max|g(x,y)+\Upsilon\lambda\int_0^y\int_0^1 k(x,y,s,t)V(f_n(x,y))\mathrm{d}s\mathrm{d}t \\
&- g(x,y)-\Upsilon\lambda\int_0^y\int_0^1 k(x,y,s,t)V(f(x,y))\mathrm{d}s\mathrm{d}t| \\
&\le \max|\lambda||\Upsilon|\int_0^y\int_0^1 |k(x,y,s,t)|\,|V(f_n(x,y))-V(f(x,y))|\,\mathrm{d}s\mathrm{d}t \\
&\le |\lambda||\Upsilon|ML\|f_n(x,y)-f(x,y)\| \\
&\Rightarrow \|f_n(x,y)-f(x,y)\| \le \beta\|f_n(x,y)-f(x,y)\|.
\end{aligned}$$

Where $\beta = L|\lambda||\Upsilon|M$. we get $(1-\beta)\|f_n(x,y)-f(x,y)\| \le 0$ and choose $0<\beta<1$ by increasing n, it implies $\|f_n(x,y)-f(x,y)\| \to 0$ as $n\to\infty$. □

5. Numerical examples

In this part, the illustrate instances are provided to indicate the accuracy and effectiveness of the technique. All of the computations have been done using the Mathematica 7 on a personal computer.

Example 5.1. *Consider the two-dimensional linear mixed Volterra-Fredholm integral equation* [25]

$$f(x,y) = x^2e^y + \left(\frac{2}{3}\right)x^3y^2 - \int_0^x\int_0^1 y^2e^{-t}f(s,t)\mathrm{d}t\mathrm{d}s.$$

Where $(x,y) \in [0,1) \times [0,1)$.

The precise solution is $f(x,y) = x^2e^y$. The solution for $f(x,y)$ is obtained by expansion method described in section (2), for $n = 1$ is collected as shown in Table 1.

Table 1. Numerical solution of Example 5.1. for $n = 1$.

(x,y)	Exact $f(x,y)$	Eroor
(.1,0)	1.001	0
(.1,.1)	0.0110517	3.34691×10^{-6}
(.1,.3)	0.0134986	3.03472×10^{-5}
(.1,.5)	0.0164872	8.22639×10^{-5}
(.1,.7)	0.0201375	1.48971×10^{-4}
(.1,.9)	0.024596	2.05545×10^{-4}

Example 5.2. *Consider the two-dimensional linear mixed Volterra-Fredholm integral equation* [26]

$$f(x,y) = x^2 + xy - (1.15)xy^4 - (1.16)xy^5 + \int_0^y\int_0^1 xys^2t^2f(s,t)\mathrm{d}s\mathrm{d}t.$$

Where $(x,y) \in [0,1) \times [0,1)$.

The exact solution is $f(x,y) = x^2 + xy$. The solution for $f(x,y)$ is obtained by expansion method described in section 2, for $n = 1$ is collected as shown in Table 2.

Table 2. Numerical solution of Example 5.2. for $n = 1$.

(x,y)	Exact $f(x,y)$	Approximation $f(x,y)$	Eroor
(0,0)	0,000	0,000	0,000
(.1,.1)	0.02	0.0199995	5.11111×10^{-7}
(0,.3)	0.09	0.09	0.000
(.1,.3)	0.12	0.119999	7.99995×10^{-7}
(0,.5)	0.25	0.25	0.000
(.1,.5)	0.3	0.299999	5.55539×10^{-7}
(0,.7)	0.49	0.49	0.000
(.1,.7)	0.56	0.56	3.11082×10^{-7}
(0,.9)	0.81	0.81	0.000
(.1,.9)	0,9	0,899999	5.99971×10^{-7}

Example 5.3. *Consider the two-dimensional nonlinear mixed Volterra-Fredholm integral equation*

$$f(x,y) = (1 - 3y - 2y^2 + 24x\cos y - \cos 2y - 3\cos y\sin y - 2y\sin 2y)/24 \int_0^y\int_0^1 (s+t)f^2(s,t)\mathrm{d}s\mathrm{d}t$$

Where $(x,y) \in [0,1) \times [0,1.)$

The exact solution is $f(x,y) = x\cos y$. The solution for $f(x,y)$ is obtained by expansin method described in section 3, for $n = 1$ is collected as shown in Table 3.

Table 3. Numerical solution of Example 5.3. for $n = 1$.

(x,y)	Exact $f(x,y)$	Approximation $f(x,y)$	Eroor
(0,.01)	0.0099995	0.0100081	8.63674×10^{-6}
(0,.21)	0.20999	0.209997	7.88799×10^{-6}
(0,.41)	0.40998	0.409985	5.77887×10^{-6}
(0,.61)	0.60997	0.609972	2.31199×10^{-6}
(0,.81)	0.80996	0.809957	2.50842×10^{-6}
(0,.91)	0.909955	0.909949	5.42443×10^{-6}

6. Conclusions

In this article, we utilized expansion technique for solving linear and nonlinear two-dimensional mixed Volterra-Fredholm integral equations of second type. This technique is very simple and involves less computation. Two-dimensional mixed Volterra-Fredholm integral equations are regularly difficult to solve analytically. In many circumstances, obtaining approximate solutions is required, this indicates that the presented method can be extended for the higher dimensional problems and other classes of integral equations such as linear and nonlinear two-dimensional Volterra integral equations and linear and nonlinear three-dimensional mixed Volterra-Fredholm integral equations.

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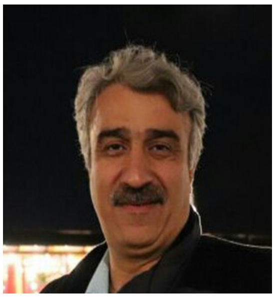

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